Thirty years of inverse problem in Hochiminh City

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July 11, 2018

Professor Dang Dinh Ang

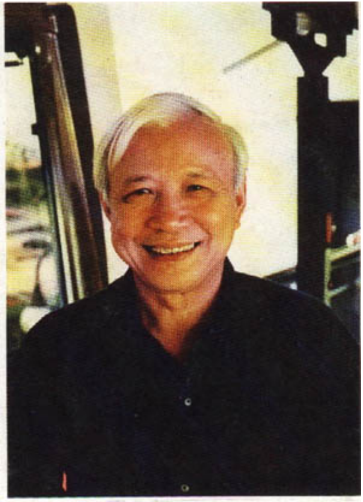


Figure: Professor Dang Dinh Ang, founder of the study of inverse problem in Hochiminh City.

Professor Dang Dinh Ang

- Professor Ang was born in 6 March 1926 in Hanoi.
- He studied Aeronautical Engineering and graduated from University of Kansas in 1955.
- He went to Caltech for graduate studies and obtained his Ph.D in Aeronautics and Mathematics in 1958.
- He is Head of The Mathematics Department of Saigon University from 1958 to 1975.
- He is Director of The Analysis Laboratory of Hochiminh University of Science from 1975 to 1995.
- He was appointed Professor of Mathematics in 1980.

Inverse problem

Input $\longrightarrow$ system $\longrightarrow$ Output

Inverse problem: From Output, Find Input or Identify the system

Ill-posed problem

Let $K : X \to Y$, $y \in Y$. Consider the problem of finding $x \in X$ such that

$$K(x) = y.$$

The problem is ill-posed in Hadamard's sense if one of three facts holds:

- The problem doesn't have any solution or
- The problem has more than two solutions or
- The problem is instability, i.e., K^{-1} is not continuous: there are $x_0, x_n \in X$, $y_0, y_n \in Y$, $K(x_n) = y_n, K(x_0) = y_0$, $y_n \to y_0$ but $x_n \not\to x_0$.

Study an ill-posed problem

- The existence of solution of the problem.
- The uniqueness of solution of the problem.
- Fix the instability of the problem: Regularization.
  If K^{-1} is not continuous, we approximate $K^{-1} : K(Y) \to X$ by a sequence of continuous operator $R_\alpha : Y \to X$ such that

$$\lim_{\alpha \to 0^+} R_\alpha(K(x)) = x \quad \forall x \in X.$$

  The operators R_α: regularization operator,
  α: regularization parameter.

Backward problem

The first paper on inverse (and ill-posed) problems in Hochiminh City is

Dang Dinh Ang, Stabilized approximate solutions of the inverse time problem for a parabolic evolution equation, J. Math. Anal. and Appl., Vol. 111 (1985), 148–155.

After that he had a survey on the problem at Banach center in Poland

Dang Dinh Ang, On the backward parabolic equation: a critical survey of some current methods in numerical analysis and mathematics modelling, Banach Center Publications, 24, Warsaw (1990), 509–515.

Backward problem

A simple example of the backward problem is of finding a function $u(x, t)$ such that

$$\begin{cases} u_t & = a u_{xx}, \quad x \in (0, \pi), \ t \in [0, T], \\ u(0, t) & = u(\pi, t) = 0, \\ u(x, T) & = g(x). \end{cases}$$

where $a > 0$, $g \in L^2(0, \pi)$. By the method of separation of variables we have

$$u(x, t) = \sum_{n=1}^{\infty} e^{a n^2 (T - t)} g_n \sin(nx),$$

where $g_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) dx$ is the Fourier coefficient of g.
The solution is instable since $e^{a n^2 (T-t)} \to \infty$ very fast for $0 \leq t < T$.
Hence, the problem is severely ill-posed.

Regularization of Backward problems

We replace the term $e^{an^2(T-t)}$ by $r(\alpha, n, t)$ such that

$$|r(\alpha, n, t)| \leq C_\alpha \quad \forall n \in \mathbb{N}, \qquad \lim_{\alpha \to 0^+} r(\alpha, n, t) = e^{an^2(T-t)},$$

and

$$R_\alpha u(x, t) = \sum_{n=1}^{\infty} r(\alpha, n, t) g_n \sin(nx).$$

For example,

- Quasi-reversibility method: $r(\alpha, n, t) = e^{an^2(T-t)/(1+\alpha n^4)}$.
- Quasi-boundary method: $r(\alpha, n, t) = \frac{e^{-an^2 t}}{\alpha + e^{-an^2 T}}$.

Some other Backward problems

Finding $u(x,t)$ such that $u(x,T) = g(x)$, $u(0,t) = u(\pi,t) = 0$ and

$$u_t = (a(u,x,t)u_x)_x + f(t,x,u),$$

$$D_t^\beta u = u_{xx} + f(t,x),$$

where $0 < \beta < 1$ and

$$D_t^\beta f = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} f'(s) ds.$$

Identification of coefficients

(with L. K. Vy) Regularized solutions of a three-dimensional inverse scattering problem for the wave equation, Inverse Problems, Vol. 8, No. 4 (1992), 499–507.

(with L. K. Vy) Coefficient identification for an inhomogeneous Helmholtz equation by asymptotic regularization, Inverse Problems, Vol. 8, No. 4 (1992), 509–523.

We consider the problem of finding the coefficient $b(x)$, $x \in \Omega \subset \mathbb{R}^3$ such that

$$\Delta u_b + b(x)u_b(x) = F(x), \;\; u_b|_{\partial\Omega} = 0$$

subject to the condition $\int_\omega \Phi(b(x))u_b(x)dx = h(x)$, where ω is open subset of Ω.

Identification of coefficients

The problem is ill-posed. Find $b \in Q \subset H_0^1(\Omega)$ by solving $J(b) = \min_{q \in Q} J(q)$ where

$$J(q) = \left| \int_\Omega \Phi(q(x)) u_q(x) dx - h(x) \right|.$$

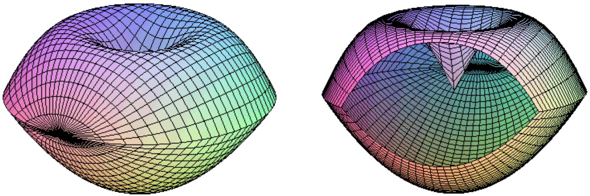
We also studied some other problems of identifying of source function $F(x,t)$ such that

$$D_t^\beta u = u_{xx} + F(x,t).$$

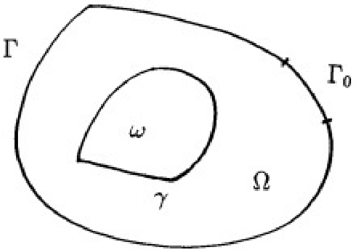
Here, $F(x,t) = \varphi(t) f(x)$.

Dang Dinh Ang, Reinhard Mennicken, Dinh Ngoc Thanh, and Dang Duc Trong, Cavity detection by the electric method: The 3-dimensional case, Z. Angew. Math. Mech., Vol. 84, No. 2 (2004), 75–85.

Let Ω be an electric conductor in $\mathbb{R}^3$. We consider the problem of identification a cavity ω inside $\Omega \subset \mathbb{R}^3$.



Identification of cavities



Let $u(x)$ be the electric potential at $x \in \Omega$ such that

$$\Delta u = 0 \text{ on } \Omega \setminus \omega, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\omega} = 0.$$

Result: the Cauchy data of u on $\Gamma \subset \partial\Omega$ defines uniquely the cavity Ω.

Identify cavities in elastic bodies

(with D. D. Trong and M. Yamamoto) Identification of cavities inside two-dimensional heterogeneous isotropic elastic bodies, Journal of Elasticity, Vol. 56 (1999), 199–212.

To prove the uniqueness we have to prove an original result

(with D. D. Trong, M. Ikehata and M. Yamamoto) Unique continuation for a stationary isotropic lam system with variable coefficients, Communications in Partial Differential Equations, Vol. 23 (1998), 599–617.

Identify cavities

(with D. D. Trong) Crack detection in plane semilinear elliptic equations in the plane: the zero flux case, Zeitschrift fur Analysis und ihre Anwendungen, Vol. 19 (2001), 109–120.

(with N. Dung, N. V. Huy and D. D. Trong) Uniqueness of elastic continuation in a semilinear elastic body, Vietnam Journal of Mechanics, Vol. 25, No. 1 (2003), 1–8.

(with D. D. Trong) Identification of cavities in a three dimensional elastic body, Zeitschrift fur Analysis und ihre Anwendungen, Vol. 23, No. 2 (2004), 407–422.

Integral equation

We consider the problem of finding x such that $Kx = y$, where K is an integral operator. The problem is ill-posed. Hence regularization is in order.

(with D. D. Hai) Regularisation of Abel's integral equation, Proceedings of the Royal Society of Edinburgh, 107A (1987), 165–168.

Consider the Abel integral operator

$$J_\alpha u(t) = \int_0^t u(s)(t-s)^{-\alpha} ds$$

where $0 < \alpha < 1$. Suppose u is in $H^1(0,1)$, and f is an element of $L^2(0,1)$ such that $\|J_\alpha u - f\|_{L^2} < \epsilon$.

Abel integral equation

The paper constructed the regularization solution

$$u_\beta(x) = u_{1\beta}(x) + u_{2\beta}(1-x),$$

where $u_{j\beta}$ satisfy

$$\beta u_{j\beta} + \int_0^t u_{j\beta}(s)ds = g_j(x)$$

where

$$\begin{aligned} g_1(x) &= \frac{\sin \pi\alpha}{\pi}\left\{xJ_{1-\alpha}f(x) - \int_0^x J_{1-\alpha}f(s)ds\right\}, \\ g_2(x) &= \frac{\sin \pi\alpha}{\pi}\left\{-xJ_{1-\alpha}f(1-x) + \int_0^x J_{1-\alpha}f(1-s)ds\right\}. \end{aligned}$$

Inverse of the Laplace transform

(with J. Lund and F. Stenger) Complex variable and regularization methods of inversion of the Laplace transform, Mathematics of Computation, Vol. 53 (1989), 589–608.

Let $g : \mathbb{R}^+ \to \mathbb{R}$. We find $f : \mathbb{R}^+ \to \mathbb{R}$ such that $\|\mathcal{L}f - g\| \le \epsilon$ where

$$\mathcal{L}f \equiv \int_0^\infty f(s)e^{-st}ds.$$

We have the Bromwich formula

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}g(s)ds.$$

However, the function g is not given on $[c - i\infty, c + i\infty]$. The problem is ill-posed.

Inverse of the Laplace transform

We can use the Tikhonov regularization method to approximate f by

$$\beta f_\beta + \mathcal{L}^* \mathcal{L} f_\beta = \mathcal{L}^* g$$

or

$$\beta f_\beta(t) + \int_0^\infty \frac{f_\beta(u)}{u+t} du = \int_0^\infty e^{st} g(s) ds.$$

We have

$$\|f_\beta - f\| \leq C\epsilon^{1/2}.$$

Moment problem

Let $\Omega \subset \mathbb{R}^d$, σ_n be a sequence of measures on $\mathbb{R}^d$. We find a function $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega u(x) d\sigma_n = \mu_n, \quad n \in \mathbb{N}.$$

If $d\sigma_n = g_n dx$, then we have the problem

$$\int_\Omega u(x) g_n(x) dx = \mu_n, \quad n \in \mathbb{N}.$$

If $d\sigma_n = \delta(x - x_n)$, then we have the problem

$$u(x_n) = \mu_n, \quad n \in \mathbb{N}.$$

Hausdorff moment problem

(with R. Gorenflo and D. D. Trong) A multidimensional Hausdorff moment problem: Regularization by finite moments, Zeitschrift fur Analysis und ihre Anwendungen, Vol. 18, No. 1 (1999), 13–25.

We find $u \in L^2(0,1)$ such that

$$\left| \int_0^1 x^n u(x) dx - \mu_n \right| \leq \epsilon \quad \forall n \in \mathbb{N}.$$

To regularize the problem, we use the orthonormalization method. Denote

$$C_{nk} = (2n+1)^{1/2} (-1)^k \frac{(n+k)!}{(n-k)!(k!)^2},$$

$$L_n(x) = \sum_{k=0}^{n} C_{nk} x^k, \ \lambda_k = \sum_{p=0}^{k} C_{np} \mu_p.$$

L_n: the Legendre polynomial.

Hausdorff moment problem

Put $\mu = (\mu_n)$,

$$p_n(\mu) = \sum_{k=0}^{n} \lambda_k x^k.$$

Then there exists $n(\epsilon)$ such that

$$\|p_{n(\epsilon)} - u\|_{L^2} \le \frac{C}{\ln \frac{1}{\epsilon}}.$$

(with L. K. Vy and D. D. Trong) Reconstruction of an analytic function from a sequence of values: existence and regularization, Finite or Infinite Dimensional Complex Analysis and Applications, 127–142, Adv. Complex Anal. Appl., 2, Kluwer Acad. Publ. 2004.

(with R. Gorenflo, Vy K. Le and D. D. Trong) Moment Theory and some Inverse problems in Potential theory and Heat conduction, Springer-Verlag, Berlin, Heidelberg, 2002.

Inverse Stefan problem

(with R. Gorenflo and D. N. Thanh) Regularization for a two dimensional inverse Stefan problem, Proceedings, International Workshop "Inverse Problems" in HoChiMinh City, January 1995, editors: Ang-Gorenflo et al. Public, No. 2, HoChiMinh Math. Soc., 1995, pp. 45–54.

(with A. Pham, N. D and D. N. Thanh) Regularization of a two-dimensional two-phase inverse Stefan problem, Inverse Problems, Vol. 13, No. 3 (1997), 607–619.

Cauchy problem

(with D. N. Thanh and V. V. Thanh) Regularized solutions of a Cauchy problem for the Laplace equation in an irregular strip, J. Integral Equations Appl., Vol. 5, No. 4 (1993), 429–441.

(with N. H. Nghia and N. C. Tam) Regularized solutions of a Cauchy problem for the Laplace equation in an irregular layer: a 3-D model, Acta. Math. Vietnam, Vol. 23 (1998), 65–74.

(with L. K. Vy) Contact of a viscoelastic body with a rough rigid surface and identification of coefficients, J. Inverse and Ill-Posed Problems, Vol. 9, No. 1 (2001), 1–20.

(with P. T. Trinh) Inverse problem in Geosciences, J. Geology., Series B, No. 13-14 (1999), 194–199.

(with D. D. Trong and M. Yamamoto) A Cauchy problem for elliptic equations: quasi-reversibility and error estimates. Vietnam J. Math., Vol. 32 (2004), 9–17.